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These notes have not received the scrutiny of publication. They could be missing important references, etc.

Down-Up Walks on Simplicial Complexes

1 Overview

- Last few classes: Glauber dynamics/Gibbs Sampler are specific to product spaces $\otimes_{i=1}^{n}\Sigma_i$, such as $\{\pm 1\}^n$.
- We want to sample important objects that don't live in product spaces. Eg. Spanning trees ina graph G. If μ is the uniform measure on spanning trees, we could view μ as a measure on $\{0,1\}^E$ where E denotes the edges. We cannot use Glauber dynamics because it doesn't do anything (it will add the deleted vertex back, so $P = I$).
- The down-up walk is a generalization of Gibbs sampler, which samples from "faces of a simplicial complex" (generalization of triangles in higher dimension). i.e. elements of $\binom{[n]}{k}$ for some k.

Notation 1. μ is a prob. measure on $\binom{[n]}{k}$ subsets of $[n]$ of size k. $k-1$ is the "dimension." Eg. a triangle is 2-dim with 3 elements and a tetrahedron is $3-$dim with 4 elements.

2 Definition

Definition 1. Down operator $D_{k\to k-1}$ acts by dropping 1 vertex uniformly at random. Then measure $(\delta_s D_{k\to k-1})_T = \frac{1}{k}$ if $T \subset S$, $|T| = k-1$, and 0 otherwise.

Definition 2. Up operator $U_{k-1\to k}$ is defined by the measure $(\delta_T U_{k-1\to k})_S = \mu(S)$ if $S \supset T, |S| = k$, and 0 otherwise.

Definition 3. Down-up walk: $P = D_{k\to k-1}, U_{k-1\to k}$

3 Lemmas

Lemma 1. *μ is stationary w.r.t. P.*

Proof. Sample $s \sim \mu$ and apply the down-up walk to obtain s'. Then

$$\mathbb{P}(s'|T) \propto \mathbf{1}_{s'\supset T}\mu(s)$$

By the law of total expectation,

$$P(s') = \mathbb{E}_{s,T}[\mathbb{P}(s'|T)] = \mathbb{E}_T[\mathbb{P}(s = s'|T)] = \mu(s')$$

□

Lemma 2. *$\forall f, g, f : \binom{[n]}{k-1} \to \mathbb{R}, g : \binom{[n]}{k} \to \mathbb{R}$, we have $\langle D_{k\to k-1} f, g\rangle_{\mu_k} = \langle f, U_{k-1\to k} g\rangle_{\mu_{k-1}}$, where $\mu_k = \mu, \mu_{k-1} = \mu D_{k\to k-1}$ is the induced measure by dropping 1 edge.*

Proof.

$$(U_{k-1\to k}g)(T) = \mathbb{E}[g(s)|s \xrightarrow{D_{k\to j-1}} T]$$
$$\langle D_{k\to k-1}f, g\rangle = \mathbb{E}_{S\xrightarrow{D_{k\to k-1}}T}[f(T)g(s)]$$
$$= \mathbb{E}[\mathbb{E}[f(T)g(s)|T]] = \mathbb{E}[f(T)\mathbb{E}[g(s)|T]] = \langle f, U_{k-1\to k}g\rangle$$

□

Lemma 3. *μ is reversible w.r.t P.*

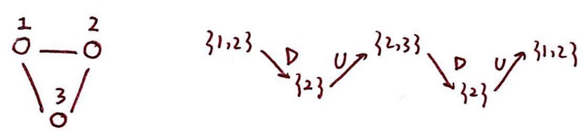
Proof. WTS $\langle Pf, g\rangle_{\mu_k} = \langle f, Pg\rangle_{\mu_k}$. Since μ is stationary w.r.t. P, using the previous lemma,

$$\langle D_{k\to k-1}U_{k-1\to k}f, g\rangle_{\mu_k} = \langle U_{k-1\to k}f, U_{k-1\to k}g\rangle_{\mu_{k-1}} = \langle f, D_{k\to k-1}U_{k-1\to k}g\rangle_{\mu_k}$$

□

4 Examples

Example 1. (Lazy SRW) Let μ_k be the uniform measure on edges of graph G, $D_{2\to 1}U_{1\to 2}$ is the down-up walk on edges, then we have the lazy simple random walk on edges. Similarly, $U_{1\to 2}D_{2\to 1}$ is the lazy SRW on vertices.



Example 2. (Glauber Dynamics) Probability measure π on $\otimes_{i=1}^{n}\Sigma_i$, $\Sigma = \{(i, x_i) : i \in [n] \text{ vertex}, x_i \in \Sigma_i\}$, $|\Sigma| = \sum_{i=1}^{n} |\Sigma_i|$. The Glauber dynamics is the down-up walk on μ.

$$\mu(\{(1, x_1), (2, x_2), \cdots, (n, x_n)\}) = \pi((x_1, \cdots, x_n))$$

Down: delete a coordinate $i \sim [n]$. Up: resample x_i given $x \sim i$ according to π. μ is the probability measure on $\binom{\Sigma}{n}$.

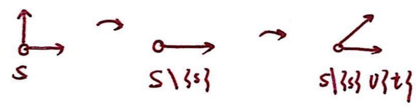
Example 3. (Spanning tree walk) $G = ([n], E)$ is a graph. $\mu(s) \propto \mathbf{1}_{s \text{ is a spanning tree of } G}$. $S \subset E$ is th edge set of G,$s \in \binom{E}{n-1.}$

Example 4. (Basis exchange walk) $\mu(s) \propto \mathbf{1}_{s \text{ is a basis of matroid } M}$. See next section for the definition of matroids.

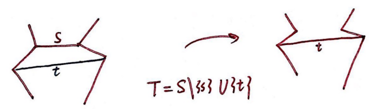
5 Matroids

Definition 4. A matroid is a pure simplicial complex (K-uniform hypergraph, subset of $\binom{[n]}{k}$)) st. if B denotes the bases (faces of complex) st. $\forall S, T \in B, \forall s \in S/T, \exists t \in T/S$ st. $(S/\{s\}) \cup \{t\} \in B$. This is called the "basis exchange property".

Example 5. (Matroids) See picture. Linear matroid vectors $\{v_1, \cdots, v_m\} \in V$, where V is a vector space. $B = \{$ bases for $span(\epsilon)\}$.



Example 6. (Spanning tree) See picture. After deleting s from S and adding t, we obtain another spanning tree.



Theorem 1. *(Anari-Liu-Oveis Gharan-Vinzant '19) Let μ be the uniform measure on basses of matroid M. Let P be the down-up walk on M. Then $1 - \lambda_2(P) = \Omega(\frac{1}{k})$, where $\Omega(\frac{1}{k})$ means it is at least of size $\frac{1}{k}$.*

Theorem 2. *Let μ be a probability measure on $\binom{[n]}{k.}$ Let $g_\mu(z) := \sum_s \mu(s) \prod_{i \in S} z_i$. If $\log g_\mu$ is concave on $\mathbb{R}^n_{>0}$, then $1 - \lambda_2(P) = \Omega(\frac{1}{k})$.*