These notes have not received the scrutiny of publication. They could be missing important references, etc.

Proof of Trickle-Down

1 Recall

1. Generating polynomial: $g_\mu(\lambda) = \sum_{S\in([n])} \mu(S) \prod_{u\in S} \lambda_u$, where $\lambda = (\lambda_1, \cdots, \lambda_n)$. Tilt: $(\mu * \lambda)(S) = \frac{1}{g_\mu(\lambda)} \mu(S) \prod_{u\in S} \lambda_u$.
2. $(\mu * \lambda)(S)$ are 1-Spectral Independent (SI) $\iff$ g_μ is log-concave $\iff$ $\nabla^2 \log g_\mu|_\lambda \preceq 0$ for any λ.
3. μ is C-SI $\iff$ $\mathrm{Cov}_\mu(\mathbb{1}_S) \preceq C \operatorname{diag}(\mathbb{E}\mathbb{1}_S) \iff \lambda_2(P_{1\leftrightarrow k}) = \lambda_2(P_{k\leftrightarrow 1}) \leq \frac{C}{k}$

2 Two untight bounds

μ is a prob. measure on $\binom{[n]}{k}, S \sim \mu, S = \{u_1, \cdots, u_k\}$. We establish two untight bounds for Covariance:

Bound 1. $\mathrm{Cov}(\mathbb{1}_S) \preceq \frac{C}{1-\lambda_2} \mathbb{E}[\operatorname{diag}(\mathbb{E}\mathbb{1}_{S\setminus u_1} \mid u_1)]$

Proof. By law of total variance:

$$\mathrm{Cov}_\mu(\mathbb{1}_S) = \mathbb{E}[\mathrm{Cov}(\mathbb{1}_S \mid u_1)] + \mathrm{Cov}(\mathbb{E}[\mathbb{1}_S \mid u_1])$$

write $\mathbb{1}_S = \mathbb{1}_{S\setminus u_1} + \mathbb{1}_{u_1}$, then $\mathrm{Cov}(\mathbb{1}_S \mid u_1) = \mathrm{Cov}(\mathbb{1}_{S\setminus u_1} \mid u_1) \preceq C \operatorname{diag} \mathbb{E}(\mathbb{1}_{S\setminus u_1} \mid u_1)$. By variational characteristics $\mathrm{Var}(U_{1\to k} f) \preceq \lambda_2(P_{1\leftrightarrow k}) \mathrm{Var}(f)$, then

$$\mathrm{Cov}(\mathbb{E}(\mathbb{1}_S \mid u_1)) = \mathrm{Cov}(U_{1\to k} \mathbb{1}_S) \preceq \lambda_2 \mathrm{Cov}(\mathbb{1}_S)$$

and hence

$$\mathrm{Cov}_\mu(\mathbb{1}_S) \preceq \mathbb{E} C \operatorname{diag} \mathbb{E}(\mathbb{1}_{S\setminus u_1} \mid u_1) + \lambda_2 \mathrm{Cov}(\mathbb{1}_S)$$

thus

$$\mathrm{Cov}(\mathbb{1}_S) \preceq \frac{C}{1-\lambda_2} \mathbb{E}[\operatorname{diag}(\mathbb{E}\mathbb{1}_{S\setminus u_1} \mid u_1)]$$

□

Bound 2. $\mathrm{Cov}(\mathbb{1}_S) \preceq C \left(I - \frac{\mathbb{1}\mathbb{1}^T}{n}\right) \operatorname{diag}(\mathbb{E}\mathbb{1}_S) \left(I - \frac{\mathbb{1}\mathbb{1}^T}{n}\right)$

Proof. Note that C-SI $\iff$ $\mathrm{Cov}(\mathbb{1}_S) \preceq C \operatorname{diag}(\mathbb{E}\mathbb{1}_S)$. $\mathbb{1}^T \mathrm{Cov}(\mathbb{1}_S) \mathbb{1} = \mathrm{Var}(\mathbb{1}_S^T \mathbb{1}_S) = 0$; and $\mathbb{1}^T C \operatorname{diag}(\mathbb{E}\mathbb{1}_S) \mathbb{1} = C \sum_i \mu(i \in S) = Ck$, then

$$\left(I - \frac{\mathbb{1}\mathbb{1}^T}{n}\right) \mathrm{Cov}(\mathbb{1}_S) \left(I - \frac{\mathbb{1}\mathbb{1}^T}{n}\right) = \mathrm{Cov}(\mathbb{1}_S)$$

and hence

$$\mathrm{Cov}(\mathbb{1}_S) \preceq C \left(I - \frac{\mathbb{1}\mathbb{1}^T}{n}\right) \operatorname{diag}(\mathbb{E}\mathbb{1}_S) \left(I - \frac{\mathbb{1}\mathbb{1}^T}{n}\right)$$

is a tighter bound. □

3 Trickle-Down

Lemma 1 (Key Lemma)**.** *μ is a prob. measure on $\binom{[n]}{k}$, $S \sim \mu$, μ is C-SI* $\iff$ $\mathrm{Cov}(\mathbb{1}_S) \preceq C \cdot \left(\mathrm{diag}(\mathbb{E}\mathbb{1}_S) - \frac{1}{k}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T\right)$.

Proof. since $f^T \mathrm{Cov}(\mathbb{1}_S) f = \mathrm{Var}(\mathbb{1}_S^T f) = \mathrm{Var}(\sum_{i \in S} f(i))$, then

$$\begin{aligned}\mathrm{Var}(\sum_{i \in S} f(i)) &= k^2 \langle f, (P - \mathbb{1}\mu_1) f \rangle_{\mu_1} \\ &= k^2(\langle f, P_f \rangle - (\mu_1 f)^2) =: (*)\end{aligned}$$

note $\mu_1 = \mu D_{k \to 1}, \mu_1(i) = \frac{1}{k}\mu(i \in S)$, $P_{ij} = \frac{1}{k}\mu(j \in S \mid i \in S)$, $(\mathrm{diag}(\mu_1)P)_{ij} = \frac{1}{k^2}\mu(i \in S, j \in S)$, and

$$(\mathrm{diag}(\mu_1)(P - \mathbb{1}\mu_1))_{ij} = \frac{1}{k^2}(\mu(i \in S, j \in S) - \mu(i \in S)\mu(j \in S))$$

since $f = f - \mathbb{E}f + \mathbb{E}f$, then

$$\begin{aligned}\langle f, Pf \rangle_{\mu_1} &= \langle \mathbb{E}f, P\mathbb{E}f \rangle_{\mu_1} + \langle f - \mathbb{E}f, P(f - \mathbb{E}f) \rangle_{\mu_1} \\ &\leq \langle \mathbb{E}f, P\mathbb{E}f \rangle_{\mu_1} + \lambda_2(P) \langle f - \mathbb{E}f, f - \mathbb{E}f \rangle_{\mu_1} \\ &= (1 - \lambda_2)(\mathbb{E}f)^2 + \lambda_2(P) \langle f, f \rangle_{\mu_1}\end{aligned}$$

then

$$\begin{aligned}(*) &= k^2(\langle f, P_f \rangle - (\mu_1 f)^2) \\ &\leq k^2[(1 - \lambda_2)(\mathbb{E}f)^2 + \lambda_2 \langle f, f \rangle_{\mu} - (\mathbb{E}f)^2] \\ &= k^2 \lambda_2(\langle f, f \rangle_{\mu_1} - \lambda_2(\mathbb{E}f)^2) \\ &= f^T C \cdot \left(\mathrm{diag}(\mathbb{E}\mathbb{1}_S) - \frac{1}{k}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T\right) f\end{aligned}$$

since $\lambda_2 = C/k$. □

Lemma 2. *μ is a prob. measure on $\binom{[n]}{k}$, $S \sim \mu$, λ_2 is second largest eigenvalue of $P = U_{1 \to k} D_{k \to 1}$, f_2 is the λ_2-eigenvector of P, then*

$$\mathrm{Cov}(\mathbb{1}_S) f_2 = k\lambda_2 \,\mathrm{diag}(\mathbb{E}\mathbb{1}_S) f_2$$

Theorem 1 (Trickle-Down)**.** *Let $\mu_{\{i\}}(\cdot) = \mu(S \backslash \{i\} = \cdot \mid i \in S)$ is the link measure at $\{i\}$ on $\binom{[n-1]}{k-1}$. Suppose*

1. *$\mu_{\{i\}}$ is C-SI for all $i \in [n]$, and*
2. *$P_{k \leftrightarrow 1}$ (or $P_{k \leftrightarrow k-1}$) is ergodic*

then μ is C'-SI for

$$C' = \frac{C(k-2)}{k-1-C}$$

Note that $C = 1$ implies $C' = 1$.

Proof. By law of total variance and the key lemma,

$$\begin{aligned}
\mathrm{Cov}(\mathbb{1}_S) &= \mathbb{E}[\mathrm{Cov}(\mathbb{1}_S|u_1)] + \mathrm{Cov}(\mathbb{E}(\mathbb{1}_S|u_1)) \\
&\preceq C(k-1)\mathbb{E}\left[\frac{1}{k-1}\,\mathrm{diag}(\mathbb{E}(\mathbb{1}_{S\backslash u_1}|u_1)) - \frac{1}{(k-1)^2}\mathbb{E}[\mathbb{1}_{S\backslash u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\backslash u_1}|u_1]^T\right] \\
&\quad + \mathrm{Cov}(\mathbb{E}(\mathbb{1}_S|u_1)) \\
&= C(k-1)\mathbb{E}\left[\frac{1}{k-1}\,\mathrm{diag}(\mathbb{E}(\mathbb{1}_{S\backslash u_1}|u_1)) - \frac{1}{k^2}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T + \frac{1}{k^2}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T\right. \\
&\quad \left. - \frac{1}{(k-1)^2}\mathbb{E}[\mathbb{1}_{S\backslash u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\backslash u_1}|u_1]^T\right] + \mathrm{Cov}(\mathbb{E}(\mathbb{1}_S|u_1)) \\
&= C(k-1)\mathbb{E}\left[\frac{1}{k-1}\,\mathrm{diag}(\mathbb{E}(\mathbb{1}_{S\backslash u_1}|u_1)) - \frac{1}{k^2}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T\right] \\
&\quad - C(k-1)\mathbb{E}\left[\frac{1}{(k-1)^2}\mathbb{E}[\mathbb{1}_{S\backslash u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\backslash u_1}|u_1]^T - \frac{1}{k^2}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T\right] \\
&\quad + \mathrm{Cov}(\mathbb{E}(\mathbb{1}_S|u_1))
\end{aligned} \tag{1}$$

since $\mathbb{1}_S = \mathbb{1}_{S\backslash u_1} + \mathbb{1}_{u_1}$, then $\mathbb{E}(\mathbb{1}_{S\backslash u_1}|u_1) = \mathbb{E}(\mathbb{1}_S|u_1) - \mathbb{1}_{u_1}$, thus

$$\begin{aligned}
\mathbb{E}[\mathbb{E}(\mathbb{1}_{S\backslash u_1}|u_1)] &= \mathbb{E}[\mathbb{E}(\mathbb{1}_S|u_1)] - \mathbb{E}[\mathbb{1}_{u_1}] = \mathbb{E}(\mathbb{1}_S) - \mathbb{E}(\mathbb{1}_{u_1}) \\
&= \frac{k-1}{k}\mathbb{E}\mathbb{1}_S
\end{aligned} \tag{2}$$

that is $\mathbb{E}(\mathbb{1}_{S\backslash u_1}) = \left(1 - \frac{1}{k}\right)\mathbb{E}\mathbb{1}_S$. Since

$$\mathbb{E}\left[\mathbb{1}_{S\backslash u_1}\mathbb{1}_{S\backslash u_1}^T\right]_{ij} = \begin{cases} \left(1 - \dfrac{2}{k}\right)\mathbb{E}[\mathbb{1}_S\mathbb{1}_S^T]_{ij}, & i \neq j \\ \left(1 - \dfrac{1}{k}\right)\mathbb{E}[\mathbb{1}_S]_i, & i = j \end{cases}$$

then

$$\begin{aligned}
\mathrm{Cov}(\mathbb{1}_{S\backslash u_1}) &= \mathbb{E}\left[\mathbb{1}_{S\backslash u_1}\mathbb{1}_{S\backslash u_1}^T\right] - \mathbb{E}[\mathbb{1}_{S\backslash u_1}]\mathbb{E}[\mathbb{1}_{S\backslash u_1}]^T \\
&= \left(1 - \frac{2}{k}\right)\mathbb{E}[\mathbb{1}_S\mathbb{1}_S^T] + \frac{1}{k}\,\mathrm{diag}\,\mathbb{E}[\mathbb{1}_S] - \left(1 - \frac{1}{k}\right)^2\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] \\
&= \left(1 - \frac{2}{k}\right)\left[\mathbb{E}[\mathbb{1}_S\mathbb{1}_S^T] - \mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T]\right] + \frac{1}{k}\,\mathrm{diag}\,\mathbb{E}[\mathbb{1}_S] \\
&\quad + \underbrace{\left(1 - \frac{2}{k}\right)\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] - \left(1 - \frac{1}{k}\right)^2\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T]}_{= -\frac{1}{k^2}\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T]} \\
&= \left(1 - \frac{2}{k}\right)\mathrm{Cov}(\mathbb{1}_S) + \frac{1}{k}\,\mathrm{diag}\,\mathbb{E}[\mathbb{1}_S] - \frac{1}{k^2}\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T]
\end{aligned}$$

by law of total variance, we have

$$\begin{aligned}
\mathrm{Cov}(\mathbb{1}_S) &= \mathbb{E}\underbrace{\mathrm{Cov}(\mathbb{1}_S|u_1)}_{= \mathrm{Cov}(\mathbb{1}_{S\backslash u_1}|u_1)} + \mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1]) \\
&= \mathbb{E}[\mathrm{Cov}(\mathbb{1}_{S\backslash u_1}|u_1)] + \mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1])
\end{aligned}$$

that is $\mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1]) = \mathrm{Cov}(\mathbb{1}_S) - \mathbb{E}[\mathrm{Cov}(\mathbb{1}_{S\setminus u_1}|u_1)]$, then

$$\begin{aligned}
\mathrm{Cov}(\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]) &= \mathrm{Cov}(\mathbb{1}_{S\setminus u_1}) - \mathbb{E}[\mathrm{Cov}(\mathbb{1}_{S\setminus u_1}|u_1)] \\
&= \left(1-\frac{2}{k}\right)\mathrm{Cov}(\mathbb{1}_S) + \frac{1}{k}\operatorname{diag}\mathbb{E}[\mathbb{1}_S] - \frac{1}{k^2}\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] - \mathbb{E}[\mathrm{Cov}(\mathbb{1}_{S\setminus u_1}|u_1)] \\
&= -\frac{2}{k}\mathrm{Cov}(\mathbb{1}_S) + \frac{1}{k}\operatorname{diag}\mathbb{E}[\mathbb{1}_S] - \frac{1}{k^2}\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] + \left[\mathrm{Cov}(\mathbb{1}_S) - \mathbb{E}[\mathrm{Cov}(\mathbb{1}_{S\setminus u_1}|u_1)]\right] \\
&= -\frac{2}{k}\mathrm{Cov}(\mathbb{1}_S) + \frac{1}{k}\operatorname{diag}\mathbb{E}[\mathbb{1}_S] - \frac{1}{k^2}\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] + \mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1]) \qquad (3)
\end{aligned}$$

by equation (2), we have $\mathbb{E}(\mathbb{1}_{S\setminus u_1}) = \left(1-\frac{1}{k}\right)\mathbb{E}\mathbb{1}_S$, then

$$\begin{aligned}
\mathrm{Cov}(\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]) &= \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]^T\right] - \mathbb{E}[\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]]\mathbb{E}[\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]^T] \\
&= \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]^T\right] - \mathbb{E}[\mathbb{1}_{S\setminus u_1}]\mathbb{E}[\mathbb{1}_{S\setminus u_1}]^T \\
&= \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]^T\right] - \left(1-\frac{1}{k}\right)^2\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] \qquad (4)
\end{aligned}$$

combine (3) and (4), we have

$$\begin{aligned}
-\frac{2}{k}&\mathrm{Cov}(\mathbb{1}_S) + \frac{1}{k}\operatorname{diag}\mathbb{E}[\mathbb{1}_S] - \frac{1}{k^2}\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] + \mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1]) \\
&= \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]^T\right] - \left(1-\frac{1}{k}\right)^2\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] \\
&= (k-1)^2\mathbb{E}\left[\frac{1}{(k-1)^2}\left[\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]^T\right] - \frac{1}{k^2}\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T]\right] \qquad (5)
\end{aligned}$$

put equation (5) into equation (1), note that by equation (2)

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{k-1}\operatorname{diag}(\mathbb{E}(\mathbb{1}_{S\setminus u_1}|u_1))\right] &= \frac{1}{k-1}\operatorname{diag}\mathbb{E}\left[\mathbb{E}(\mathbb{1}_{S\setminus u_1}|u_1)\right] \\
&= \frac{1}{k-1}\operatorname{diag}\mathbb{E}(\mathbb{1}_{S\setminus u_1}) = \frac{1}{k}\operatorname{diag}\mathbb{E}(\mathbb{1}_S)
\end{aligned}$$

then

$$\begin{aligned}
\mathrm{Cov}(\mathbb{1}_S) \preceq\; & C(k-1)\mathbb{E}\left[\frac{1}{k-1}\operatorname{diag}(\mathbb{E}(\mathbb{1}_{S\setminus u_1}|u_1)) - \frac{1}{k^2}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T\right] \\
& - C(k-1)\mathbb{E}\left[\frac{1}{(k-1)^2}\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]\mathbb{E}[\mathbb{1}_{S\setminus u_1}|u_1]^T - \frac{1}{k^2}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T\right] \\
& + \mathrm{Cov}(\mathbb{E}(\mathbb{1}_S|u_1)) \\
=\; & C(k-1)\left[\frac{1}{k}\operatorname{diag}\mathbb{E}(\mathbb{1}_S) - \frac{1}{k^2}(\mathbb{E}\mathbb{1}_S)(\mathbb{E}\mathbb{1}_S)^T\right] \\
& - \frac{C}{k-1}\left[-\frac{2}{k}\mathrm{Cov}(\mathbb{1}_S) + \frac{1}{k}\operatorname{diag}\mathbb{E}[\mathbb{1}_S] - \frac{1}{k^2}\mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] + \mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1])\right] \\
& + \mathrm{Cov}(\mathbb{E}(\mathbb{1}_S|u_1))
\end{aligned}$$

rearrange it, we have

$$\begin{aligned}
\left[1 - \frac{2C}{k(k-1)}\right] \mathrm{Cov}(\mathbb{1}_S) \preceq & \left(1 - \frac{C}{k-1}\right) \mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1]) \\
& + \frac{Ck}{k-1} \operatorname{diag} \mathbb{E}[\mathbb{1}_S] - C\frac{k-2}{k(k-1)} \mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] \\
\preceq & \left(1 - \frac{C}{k-1}\right) \mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1]) \\
& + \frac{Ck}{k-1} \operatorname{diag} \mathbb{E}[\mathbb{1}_S] - C\frac{k-2}{k(k-1)} \mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T]
\end{aligned}$$

since $\mathrm{Cov}(\mathbb{E}[\mathbb{1}_S|u_1]) \preceq \lambda_2 \mathrm{Cov}(\mathbb{1}_S)$, then apply Lemma 2: $f_2^T \mathrm{Cov}(\mathbb{1}_S) f_2 = k\lambda_2 f_2^T \operatorname{diag}(\mathbb{E}\mathbb{1}_S) f_2$, we have

$$\begin{aligned}
\left[1 - \frac{2C}{k(k-1)}\right] k\lambda_2 f_2^T \operatorname{diag}(\mathbb{E}\mathbb{1}_S) f_2 = & \left[1 - \frac{2C}{k(k-1)}\right] f_2^T \mathrm{Cov}(\mathbb{1}_S) f_2 \\
\leq & \left(1 - \frac{C}{k-1}\right) \lambda_2 f_2^T \mathrm{Cov}(\mathbb{1}_S) f_2 \\
& + \frac{Ck}{k-1} f_2^T \operatorname{diag} \mathbb{E}[\mathbb{1}_S] f_2 - C\frac{k-2}{k(k-1)} f_2^T \mathbb{E}[\mathbb{1}_S]\mathbb{E}[\mathbb{1}_S^T] f_2 \\
\leq & C\frac{k-2}{k-1} f_2^T \operatorname{diag} \mathbb{E}[\mathbb{1}_S] f_2 + \left(1 - \frac{C}{k-1}\right) \lambda_2^2 f_2^T \operatorname{diag} \mathbb{E}[\mathbb{1}_S] f_2
\end{aligned}$$

that is

$$(k-1-C)\,\lambda_2^2 - (k(k-1) - 2C)\lambda_2 + C(k-2) \geq 0$$

assume that $\lambda_2 < 1$, then we have

$$\lambda_2 \leq \frac{C(k-2)}{k-1-C} \cdot \frac{1}{k}$$

□