These notes have not received the scrutiny of publication. They could be missing important references, etc.

Lecture 4 : Solving the REM via Replica Method

1 Model and Definitions

Definition 1 (Random Energy Model)**.** Let $n \in \mathbb{N}$ and define the configuration space $\mathcal{X} = \{\pm 1\}^n$. For each $X \in \mathcal{X}$, let $E(X)$ be an independent random variable distributed as

$$E(X) \sim \mathcal{N}\Big(0, \frac{n}{2}\Big).$$

For $\beta > 0$ (inverse temperature), define the probability measure p_β on $\mathcal{X}$ by

$$p_\beta(X) = \frac{1}{Z_\beta} \exp(-\beta E(X)).$$

Remark 1. Because the random variable $E(X)$ is symmetric in distribution, i.e $E(X) \sim -E(X)$, we can write:

$$p_\beta(X) = \frac{1}{Z_\beta} \exp(\beta E(X)).$$

The goal is to compute

$$\mathbb{E}[\log Z_\beta] \quad , \quad Z_\beta = \sum_{X \in \mathcal{X}} \exp(\beta E(X)).$$

Let $\{\xi(X)\}_{X \in \mathcal{X}}$ be i.i.d. standard Gumbel random variables with mean γ. Using the Gumbel trick :

$$\mathbb{E}\big[\log Z_\beta\big] = \mathbb{E}_{E,\xi}\Big[\max_{X \in \mathcal{X}} \{\beta E(X) + \xi(X)\}\Big] - \gamma.$$

By Jensen's inequality,

$$\underbrace{\mathbb{E}[\log Z_\beta]}_{\text{quenched}} \le \underbrace{\log \mathbb{E}[Z_\beta]}_{\text{annealed}}.$$

2 Log as a Limit and Replica Identity

Recall that

$$\int x^p \, dx = \begin{cases} \dfrac{x^{p+1}}{p+1} + C, & \text{if } p \neq -1 \text{ (power rule)}, \\[2ex] \log(x) + C, & \text{if } p = -1. \end{cases}$$

Informally, we can view $\log x$ as the limiting case of the power rule. Set $k := p + 1$ (so $k \to 0$ as $p \to -1$). Then

$$\frac{x^{p+1} - 1}{p+1} = \frac{e^{k \log x} - 1}{k}.$$

Use the Taylor series of the exponential around 0:

$$e^{k \log x} \approx 1 + k \log x.$$

Therefore,

$$\lim_{k\to 0}\frac{x^k-1}{k}=\lim_{k\to 0}\frac{e^{k\log x}-1}{k}=\log x.$$

This can also be derived by

$$x^k=e^{k\log x}\quad\Rightarrow\quad\frac{d}{dk}x^k=e^{k\log x}\log x=x^k\log x.$$

Evaluating at $k=0$ gives

$$\frac{d}{dk}x^k\bigg|_{k=0}=\log x,$$

which is consistent with the limit representation above.

Lemma 1. *Let X be a well-behaved random variable. Then*

$$\mathbb{E}[\log X]=\lim_{k\to 0}\frac{1}{k}\log\mathbb{E}[X^k].$$

Proof. Use the identity $X^k=e^{k\log X}$:

$$\mathbb{E}[X^k]=\mathbb{E}[e^{k\log X}]\iff\mathbb{E}[X^k]=\frac{1}{k}\log\mathbb{E}[X^k].$$

Since $\mathbb{E}[e^{k\log X}]$ is differentiable at $k=0$ and $\mathbb{E}[e^{0\cdot\log X}]=1$, we have

$$\frac{d}{dk}\log\mathbb{E}[X^k]\bigg|_{k=0}=\frac{\mathbb{E}[X^k\log X]}{\mathbb{E}[X^k]}\bigg|_{k=0}=\mathbb{E}[\log X].$$

Hence,

$$\mathbb{E}[\log X]=\lim_{k\to 0}\frac{1}{k}\log\mathbb{E}[X^k].$$

□

We now apply lemma 1 to Z_β:

$$\mathbb{E}[\log Z_\beta]=\lim_{k\to 0}\frac{1}{k}\log\mathbb{E}[Z_\beta^k].$$

Therefore, it remains to compute $\mathbb{E}[Z_\beta^k]$.

3 Replica Method: Integer Moments

The key idea of the *replica method* is to first compute $\mathbb{E}[Z_\beta^k]$ for integer $k\geq 1$, and then to extend k from the integers to real values near 0.

For notation purposes, let $Z:=Z_\beta$.

3.1 Remark on the Moment Problem

In general, the moments $\mathbb{E}[Z^k]$ for $k \geq 0$ do not always uniquely determine the distribution (this is the *moment problem*). It is related to existence of the moment generating function (see Carleman's condition). For example:

$$f(x) = e^{-|x|} \quad \text{has } M_X(t) < \infty \text{ for } t \text{ in a neighborhood of } 0,$$

whereas

$$f(x) = e^{-|x|^{0.99}} \quad \text{has } M_X(t) = \infty \text{ for all } t > 0 \quad \text{(tails are too heavy).}$$

Likewise, the former distribution is determined by its moments but the latter is not. In the context of the replica method, Z is a sum of terms of the form e^G, with G a gaussian. Therefore Z does not have a moment-generating function, since

$$\mathbb{E}[e^{ke^G}] = \infty \quad \text{for any } k > 0$$

Hence, the values of $\mathbb{E}[Z^k]$ at integer k likely do not determine the distribution uniquely (but in practice, the replica method works nonetheless).

3.2 Replica Expression

We are interested in the limit

$$\lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[\log Z] = \lim_{n\to+\infty} \lim_{k\to 0} \frac{1}{n} \cdot \frac{1}{k} \log \mathbb{E}[Z^k]. \tag{1}$$

We first compute for integer $k \geq 1$:

$$\mathbb{E}[Z^k] = \mathbb{E}\left[\left(\sum_{X\in\mathcal{X}} \exp(\beta E(X))\right)^k\right] = \sum_{X_1,\ldots,X_k\in\mathcal{X}} \mathbb{E}\Big[\exp\big(\beta(E(X_1) + \cdots + E(X_k))\big)\Big].$$

Let $\mathbf{E} = \big(E(X_1), E(X_2), \ldots, E(X_k)\big)^\top$. By assumption, the random variables $\{E(X_i)\}_{i=1}^k$ are independent, centered Gaussian random variables. Hence, their covariance matrix $\mathbf{\Sigma} \in \mathbb{R}^{k\times k}$ is diagonal such that

$$\mathbf{\Sigma}_{ij} = \mathbb{E}\big[E(X_i)E(X_j)\big] = \frac{n}{2}\,\mathbf{1}\{X_i = X_j\}.$$

For any $\mathbf{a} \in \mathbb{R}^k$, the moment generating function is:

$$\mathbb{E}\big[\exp(\mathbf{a}^\top \mathbf{E})\big] = \exp\left(\frac{1}{2}\,\mathbf{a}^\top \mathbf{\Sigma}\,\mathbf{a}\right).$$

Taking $\mathbf{a} = \beta\,\mathbf{1}_k$, we obtain:

$$\begin{aligned}
\mathbb{E}[Z^k] &= \sum_{X_1,\ldots,X_k\in\mathcal{X}} \mathbb{E}\big[\exp(\mathbf{a}^\top \mathbf{E})\big] \\
&= \sum_{X_1,\ldots,X_k\in\mathcal{X}} \exp\left(\frac{1}{2}\,\mathbf{a}^\top \mathbf{\Sigma}\,\mathbf{a}\right) \\
&= \sum_{X_1,\ldots,X_k\in\mathcal{X}} \exp\left(\frac{\beta^2}{2} \sum_{i,j=1}^k \mathbb{E}\big[E(X_i)E(X_j)\big]\right) \\
&= \sum_{X_1,\ldots,X_k\in\mathcal{X}} \exp\left(\frac{\beta^2 n}{4} \sum_{i,j=1}^k \mathbf{1}\{X_i = X_j\}\right).
\end{aligned}$$

4 Overlap Representation and Counting

4.1 Overlap matrix and partitions

Notice that we sum over all k-tuples $(X_1, \ldots, X_k)$, where each $X_i \in \mathcal{X} = \{\pm 1\}^n$. Two elements X_i and X_j may coincide or differ, and the expression

$$\sum_{i,j=1}^{k} \mathbf{1}\{X_i = X_j\}$$

depends only on which configurations are the same.

We introduce the *overlap matrix*

$$\mathbf{Q}_{ij} = \mathbf{1}\{X_i = X_j\}, \quad 1 \leq i, j \leq k.$$

$\mathbf{Q}$ defines an equivalence relation on $\{1, \ldots, k\}$:

$$i \sim j \quad \text{iff} \quad X_i = X_j \;\; \text{(i.e. same on all } n \text{ entries)}.$$

Let the number of distinct configurations among $(X_1, \ldots, X_k)$ be r. Equivalently, the equivalence relation induced by $\mathbf{Q}$ partitions the index set $\{1, \ldots, k\}$ into r disjoint subsets:

$$\{A_1, \ldots, A_r\}, \qquad A_1 \sqcup A_2 \sqcup \cdots \sqcup A_r = \{1, \ldots, k\}.$$

Note that to choose r distinct configurations from $\mathcal{X}$ (of size 2^n), there are $\binom{2^n}{r}$ possible choices.

Let $\mathcal{Q}_r$ denote the set of all possible overlap matrices $\mathbf{Q}$ that correspond to partitions with exactly r distinct equivalence classes. Thus we can write:

$$\mathbb{E}[Z^k] = \sum_{r=1}^{k} \binom{2^n}{r} \sum_{Q \in \mathcal{Q}_r} \exp\left(\frac{\beta^2 n}{4} \sum_{i,j=1}^{k} Q_{ij} \right).$$

5 Asymptotic Evaluation as $n \to \infty$

5.1 Large-n counting and overlap sizes

Assume that as $n \to +\infty$, the quantity

$$\frac{1}{n} \cdot \frac{1}{k} \log \mathbb{E}[Z^k]$$

is continous in the variable k at $k = 0$. We can then interchange limits in ($\star$):

$$\lim_{n \to +\infty} \frac{1}{n} \mathbb{E}[\log Z] = \lim_{k \to 0} \lim_{n \to +\infty} \frac{1}{n} \cdot \frac{1}{k} \log \mathbb{E}[Z^k].$$

Let $a_1, \ldots, a_r$ denote the sizes of the equivalence classes (so that $a_1 + \cdots + a_r = k$). Then:

$$\sum_{i,j=1}^{k} Q_{ij} = \sum_{j=1}^{r} a_j$$

Also note that as $n \to +\infty$:

$$\binom{2^n}{r} \approx \exp(nr \log 2).$$

Therefore, asymptotocally we can write:

$$\mathbb{E}[Z^k] \approx \sum_{r=1}^{k} \exp\left[n \left(r \log 2 + \frac{\beta^2}{4} \sum_{j=1}^{r} a_j^2 \right) \right].$$

5.2 Dominant exponential

Finally, since the sum is dominated by its largest exponential term as $n \to +\infty$, we can replace the summation by the maximizer over both the number of equivalence classes r and the admissible $(a_1, \dots, a_r)$:

$$\frac{1}{k} \cdot \frac{1}{n} \log \mathbb{E}[Z^k] \longrightarrow \frac{1}{k} \max_{\substack{1 \le r \le k \\ a_1 + \dots + a_r = k}} \left(r \log 2 + \frac{\beta^2}{4} \sum_{j=1}^{r} a_j^2 \right). \qquad (\star)$$

Observe that $\sum_{j=1}^{r} a_j^2$ is maximized when the mass is most uneven (one large block) and is minimized when the blocks are perfectly balanced. Hence, the two extreme cases are:

$$\text{(i) } r = 1 : \quad \sum a_j^2 = k^2,$$

$$\text{(ii) } r = k : \quad \sum a_j^2 = k.$$

We want a trade-off between the terms:

- $r \log 2$: increases with r;
- $\frac{\beta^2}{4} \sum_{j=1}^{r} a_j^2$: decreases with r.

We approximate the balanced case by setting

$$a_j = \frac{k}{r}, \qquad j = 1, \dots, r.$$

Then

$$\sum_{j=1}^{r} a_j^2 = r \left(\frac{k}{r} \right)^2 = \frac{k^2}{r}.$$

Substituting into $(\star)$ gives the asymptotic approximation

$$\frac{1}{k} \cdot \frac{1}{n} \log \mathbb{E}[Z^k] \approx \frac{1}{k} \max_{1 \le r \le k} \left(r \log 2 + \frac{\beta^2}{4} \frac{k^2}{r} \right).$$

6 Optimization over r

Now, we wish to maximize the function

$$f(r) = r \log 2 + \frac{\beta^2 k^2}{4r},$$

with derivatives

$$f'(r) = \log 2 - \frac{\beta^2 k^2}{4r^2} \quad , \quad f''(r) = \frac{\beta^2 k^2}{2r^3} > 0.$$

Therefore, f is a convex function and the maximization is over the closed interval $[1, k]$. The maxima happen at the boundaries:

- $f(1) = \log 2 + \frac{\beta^2 k^2}{4}$.
- $f(k) = k \log 2 + \frac{\beta^2 k}{4}$.

Hence, for $k \ge 1$, the asymptotic expression in $(\star)$ gives

$$\frac{1}{k} \cdot \frac{1}{n} \log \mathbb{E}[Z^k] \approx \max \left\{ \frac{1}{k} \left(\log 2 + \frac{\beta^2 k^2}{4} \right), \log 2 + \frac{\beta^2}{4} \right\}$$

7 Replica Limit

7.1 Taking $k \to 0$

Recall that as part of the replica method, we now consider $k < 1$. We guess that instead of taking the maximum of f, we should take the minimum. Setting $f'(r) = 0$, we get :

$$r^* = \frac{\beta k}{2\sqrt{\log 2}} \quad , \quad f^* = f(r^*) = \beta\sqrt{\log 2}.$$

Therefore, we have 3 options :

$$\frac{1}{n}\mathbb{E}[\log Z] \approx \min\left\{\frac{1}{k}\left(\log 2 + \frac{\beta^2 k^2}{4}\right),\ \log 2 + \frac{\beta^2}{4},\ \beta\sqrt{\log 2}\right\}.$$

Note that :

$$\frac{1}{k}\left(\log 2 + \frac{\beta^2 k^2}{4}\right) \to +\infty \quad , \quad k \to 0.$$

The two remaining expressions coincide at the critical point β_c

$$\log 2 + \frac{\beta_c^2}{4} = \beta_c\sqrt{\log 2} \quad \Longrightarrow \quad \beta_c = 2\sqrt{\log 2}.$$

Hence,

$$\frac{1}{n}\mathbb{E}[\log Z] = \begin{cases} \log 2 + \dfrac{\beta^2}{4}, & \text{if } \beta < \beta_c \quad \text{(high temperature)}, \\ \beta\sqrt{\log 2}, & \text{if } \beta \geq \beta_c \quad \text{(low temperature)}. \end{cases}$$

8 Consistency Checks

8.1 Jensen bound

By Jensen's inequality,

$$\frac{1}{n}\mathbb{E}[\log Z] \leq \frac{1}{n}\log \mathbb{E}[Z] = \log 2 + \frac{\beta^2}{4},$$

which is the high-temperature case.

8.2 Slope check

Differentiating

$$\frac{1}{n}\log Z = \frac{1}{n}\log \sum_{X\in\mathcal{X}} \exp(\beta E(X))$$

with respect to β gives

$$\frac{d}{d\beta}\left(\frac{1}{n}\log Z\right) = \frac{1}{n}\frac{\sum_{X\in\mathcal{X}} E(X)\exp(\beta E(X))}{\sum_{X\in\mathcal{X}}\exp(\beta E(X))} = \frac{1}{n}\mathbb{E}[E(X)].$$

Since $E(X) \sim \mathcal{N}(0, n/2)$, we have

$$\mathbb{E}[E(X)] \leq \max_{X\in\mathcal{X}} E(X) \approx n\sqrt{\log 2}.$$

This is consistent the slope $\sqrt{\log 2}$ of the low-temperature case.