These notes have not received the scrutiny of publication. They could be missing important references, etc.

Lecture 8: Random Matrices II

Remark 1 (Course logistics)**.** Project proposal is due on Nov. 7th.

Remark 2 (Lecture notes)**.** These lecture notes are a continuation of the previous session, so you may read the earlier notes beforehand.

1 Top Eigenvalue of a GOE Matrix (Continued)

Let J be a GOE matrix. Recall for

$$\lambda_{\max}(J) = \max_{x \in S^{n-1}} \langle x, Jx \rangle$$

that Poincaré inequality implies $\mathrm{Var}(\lambda_{\max}) = o(1)$ as $n \to \infty$. Last time we applied the replica trick to deduce that

$$\mathbb{E}\left[\max_{x \in S^{n-1}} \frac{1}{2}\langle x, Jx \rangle\right] = \lim_{\beta \to \infty} \frac{1}{\beta n} \mathbb{E}\left[\log Z\right] = \lim_{\beta \to \infty} \lim_{k \to 0} \frac{1}{\beta n k} \log \mathbb{E}\left[Z^k\right], \tag{1}$$

where we define

$$Z := \int_{\mu} \exp\left(\frac{\beta}{2}\langle x, Jx \rangle\right)$$

and denote μ by the uniform distribution on $\sqrt{n}S^{n-1}$. Assuming that the **replica symmetric ansatz** introduced last time holds, we have for any positive integer k that

$$\lim_{n \to \infty} \frac{1}{nk} \log \mathbb{E}\left[Z^k\right] = \frac{1}{k} \max_{q \in [0,1]} \left[\frac{\beta^2 k}{4} + \frac{\beta^2 k(k-1)q^2}{4} + \frac{1}{2}\sum_{i=1}^{k} \log \lambda_i(Q)\right], \tag{2}$$

where

$$Q = \begin{bmatrix} 1 & q & \cdots & q \\ q & 1 & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ q & q & \cdots & 1 \end{bmatrix}$$

is a $k \times k$ matrix. Our ultimate goal is to derive an asymptotic fromula for (1) as $n \to \infty$. To this end, we assume all the above limits to be exchangeable and aim to **minimize** (will be explained) the objective (2) with respect to $q \in [0, 1]$, in the regime that $k \to 0$. Observe that the eigenvalues of Q are $1 - q$ (with multiplicity $k - 1$) and $1 + (k - 1)q$ (with multiplicity 1). That is, we have

$$\lim_{n \to \infty} \lim_{k \to 0} \frac{1}{nk} \log \mathbb{E}\left[Z^k\right] = \lim_{k \to 0} \lim_{n \to \infty} \frac{1}{nk} \log \mathbb{E}\left[Z^k\right] = \lim_{k \to 0} \frac{1}{k} \min_{q \in [0,1]} f(q),$$

where

$$f(q) = \frac{\beta^2 k}{4} + \frac{\beta^2 k(k-1)q^2}{4} + \frac{1}{2}\log(1 + (k-1)q) + \frac{k-1}{2}\log(1-q). \tag{3}$$

Differentiate f with respect to q:

$$\begin{aligned} f'(q) &= \frac{\beta^2 k(k-1)q}{2} + \frac{k-1}{2}\left(\frac{1}{1+(k-1)q} - \frac{1}{1-q}\right) \\ &= \frac{k(k-1)q}{2}\left(\beta^2 - \frac{1}{(1+(k-1)q)(1-q)}\right). \end{aligned}$$

As a remark, the reason the maximization problem turns into a minimization problem lies in the fact that the sign of $(k-1)$ flips as $k \to 0$. The solutions to $f'(q) = 0$ are always either $q = 0$ or the two roots of a certain quadratic equation, which converge to $q = 1 \pm \frac{1}{\beta}$, as $k \to 0$. Hence, the optimal value of q is given by

$$q = q^* = \left(1 - \frac{1}{\beta}\right)_+.$$

Plugging back into the above formula (3), we have

$$\begin{aligned} \lim_{k\to 0} \frac{f(q^*)}{k} &= \frac{\beta^2}{4}\left(1 - \left(1 - \frac{1}{\beta}\right)^2\right) + \lim_{k\to 0} \frac{1}{2k}\left[\log\left(1 + (k-1)\left(1 - \frac{1}{\beta}\right)\right) + (k-1)\log\left(\frac{1}{\beta}\right)\right] \\ &= \frac{\beta}{2} - \frac{1}{4} + \lim_{k\to 0} \frac{1}{2}\left[\frac{1 - 1/\beta}{1 + (k-1)(1-1/\beta)} + \log\left(\frac{1}{\beta}\right)\right] \qquad \text{(by L'Hôpital's rule)} \\ &= \beta - \frac{3}{4} - \frac{1}{2}\log\beta, \end{aligned}$$

provided that $\beta \geq 1$. Otherwise, $q^* = 0$ and the limit is simply $\frac{\beta^2}{4}$. However, recall that we are currently interested in the case $\beta \to \infty$. We conclude that

$$\lim_{n\to\infty} \mathbb{E}\lambda_{\max}(J) = \lim_{\beta\to\infty} \frac{2}{\beta}\left[\beta - \frac{3}{4} - \frac{1}{2}\log\beta\right] = 2.$$

Strictly speaking, this result is just a conjecture (or a guess) as we are missing some justifications, and we will look into a rigorous proof in the next lecture.

Remark 3 (Meaning of the optimal q^*)**.** Consider $p_\beta(x) \propto \exp\left(\beta\langle x, Jx\rangle/2\right)$. If $x, y \overset{iid}{\sim} p_\beta$, then $\langle x, y\rangle/n = \pm q^* + o_p(1)$, as $n \to \infty$.